

# Riemann Extension of the Anti- Mach Space time

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## Abstract

Riemann extension for the anti Mach metric is derived, the solution of geodesic equations for the extended space are given, some properties for the extended space was studied and compared with the basic space and the constructions of a translation surface for the anti Mach metric in four dimension is established.

## 1 Introduction

Riemann extension is defined as a product of a non- Riemannian  $n$ - space and a vector  $n$ -space  $A^n$  (also called Riemannian  $2n$  space) [1]. It is possible by means of such extension, to relate the properties of a non - Riemannian  $n$  space with those of certain Riemannian  $2n$  spaces [2], [3].

For  $n$  dimensional Riemannian space of the coordinates  $x^i$ ,  $i = 1 : n$  the metric is given by:

$${}^n ds^2 = g_{ij} dx^i dx^j. \quad (1.1)$$

The Riemann extension of an affine connected space  $A^n$  is given by the canonical form [3]

$${}^{2n} ds^2 = -2\Gamma_{ij}^k \Psi_k dx^i dx^j + 2d\Psi_k dx^k, \quad (1.2)$$

where  $\Psi_k$ ,  $k = n + 1, \dots, 2n$  are the coordinates of additional space and  $\Gamma_{ij}^k = \Gamma_{ji}^k$  are the connection coefficients of the space  $A^n$ .

The geodesic equations of the metric (1.2) are given by:

$$\ddot{y}^k + \hat{\Gamma}_{ij}^k \dot{y}^i \dot{y}^j = 0, \quad (1.3)$$

where  $y^k$  are the coordinates of the extended space,  $\hat{\Gamma}_{ij}^k$  are the christoffel symbols of the second kind for the metric (1.2) and dot means differentiation

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with respect to the affine parameter  $s$ .

Dryuma [4]-[6] studied solutions of the geodesic equations in the extended Riemannian space and gives some solutions of the Einstein equations for the eight dimensional Riemann extension of the classical Schwarchild four dimensional metrics, Minkowsky space time metric in rotating coordinate system, and some properties of the Gödel space metric and its Riemann extension. We study the Riemann extension of the Anti Mach metric space time and its geodesic equations in eight dimension space with its complete solutions. In §2 we introduce the Anti Mach space time and its Riemann extension. The complete solution of the geodesic equations in extended space time is given in §3. The constructions of translation surface for the Anti Mach metric in four dimension is established in §4.

## 2 The Riemann Extension of the Anti-Mach Space time

The line element for the anti- Mach space time is given by [7]:

$$^4ds^2 = dx^2 - 4tdxdz + 2dydz + 2t^2dz^2 + dt^2. \quad (2.1)$$

The non vanishing christoffel symbols of second kind for the line element (2.1) are given by:

$$\Gamma_{13}^4 = 1, \quad \Gamma_{14}^2 = -1, \quad \Gamma_{33}^4 = -2t, \quad \Gamma_{34}^1 = -1. \quad (2.2)$$

From equations (2.1), (2.2) and (1.2) the extended metric takes the form:

$$\begin{aligned} ^8ds^2 &= 4Pdzdt + 4Qdxdt - 4Vdxdz + 4tVdz^2 \\ &\quad + 2dxdP + 2dydQ + 2dzdU + 2dtdV, \\ &= \hat{g}_{ij}dy^i dy^j \end{aligned} \quad (2.3)$$

where  $\Psi_k = (P, Q, U, V)$  are the additional coordinates.

In the local coordinates  $y^i = (x, y, z, t, P, Q, U, V)$  of this type of metrics are metrics with vanishing curvature invariant [7], [8]. They play an important role in general theory of relativity, in particular for the pp-waves [9]. The eight dimensional space (2.3) is also Einstein space with the condition that the Ricci tensor is vanishing. i.e

$$^8R_{ik} = 0 \quad (2.4)$$

From equation (1.3) the complete system of geodesic equations for the metric (2.3) read as:

$$\ddot{x} = 2\dot{z}\dot{t}, \quad (2.5)$$

$$\ddot{y} = 2\dot{x}\dot{t}, \quad (2.6)$$

$$\ddot{z} = 0 \quad (2.7)$$

$$\ddot{t} = 2t\dot{z}^2 - 2\dot{x}\dot{z}, \quad (2.8)$$

$$\ddot{P} = 4Q\dot{z}(\dot{x} - t\dot{z}) - 2\dot{t}\dot{Q} + 2\dot{z}\dot{V} \quad (2.9)$$

$$\ddot{Q} = 0 \quad (2.10)$$

$$\ddot{U} = 4P\dot{z}(\dot{x} - t\dot{z}) - 2\dot{t}\dot{P} + 2(\dot{x} - 2t\dot{z})\dot{V} \quad (2.11)$$

$$\ddot{V} = 2V\dot{z}^2 - 4Q\dot{z}\dot{t} - 2\dot{z}\dot{P} - 2\dot{x}\dot{Q}, \quad (2.12)$$

where dot denotes differentiation with respect to the parameter  $s$ .

The four equations (2.5) - (2.8) constitute the system of geodesics equations of the basic space and not contain the coordinates  $\Psi_k$  while the other four equations (2.9)- (2.12) represent a system of second order ordinary differential equations for the coordinates  $\Psi_k$  and the first derivative of the coordinates  $(x, y, z, t)$ . Solutions of the equations (2.5)- (2.12) will be given in the next section.

### 3 Solutions of the Geodesics Equations

To solve the geodesic equations (2.5) - (2.12) we take an arbitrary initial point  $x^i(s)|_{s=0} = x_0^i$  and  $\Psi_k(s)|_{s=0} = \Psi_0 = (P_0, Q_0, U_0, V_0)$  with the initial directions  $\frac{dx^i}{ds}|_{s=0} = \xi^i$  and  $\frac{d\Psi_k}{ds}|_{s=0} = h_k$ , with the supplementary condition for the geodesics

$$\hat{g}_{ij} \frac{dy^i}{ds} \frac{dy^j}{ds} = \epsilon, \quad (3.1)$$

where  $\epsilon = 0, 1, -1$  according to null, timelike and spacelike geodesics respectively, that is

$$4P\dot{z}\dot{t} + 4Q\dot{x}\dot{t} - 4V\dot{x}\dot{z} + 4tV\dot{z}^2 + 2\dot{x}\dot{P} + 2\dot{y}\dot{Q} + 2\dot{z}\dot{U} + 2\dot{t}\dot{V} = \epsilon. \quad (3.2)$$

Solutions of the first four equations (2.5)- (2.8) are given by Ozsváth and Schücking [7] and read as:

For  $\xi^3 \neq 0$

$$x = x_0 + (2t_0\xi^3 - \xi^1)s + \frac{\xi^4}{\xi^3}(1 - \cos(\sqrt{2}\xi^3)s)$$

$$+ \frac{\sqrt{2}(\xi^1 - t_0 \xi^3)}{\xi^3} \sin(\sqrt{2}\xi^3)s \quad (3.3)$$

$$\begin{aligned} y = y_0 + [\xi^2 + \frac{(\xi^4)^2 + 2(\xi^1)^2}{2\xi^3} + t_0(3t_0\xi^3 - 4\xi^1)]s + \\ \frac{\xi^4}{2(\xi^3)^2} [2(\xi^1 - 2t_0\xi^3) \cos \sqrt{2}\xi^3 s - (\xi^1 - t_0\xi^3) \cos 2\sqrt{2}\xi^3 s \\ - (\xi^1 - 3t_0\xi^3)] - \frac{\sqrt{2}}{(\xi^3)^2} [(\xi^1)^2 - t_0\xi^3(3\xi^1 - 2t_0\xi^3)] \sin \sqrt{2}\xi^3 s \\ + \frac{1}{2\sqrt{2}(\xi^3)^2} [(\xi^1)^2 - \frac{(\xi^4)^2}{2} - t_0\xi^3(2\xi^1 - t_0\xi^3)] \sin 2\sqrt{2}\xi^3 s \end{aligned} \quad (3.4)$$

$$z = z_0 + \xi^3 s \quad (3.5)$$

$$t = t_0(2 - \cos \sqrt{2}\xi^3 s) - \frac{\xi^1}{\xi^3}(1 - \cos \sqrt{2}\xi^3 s) + \frac{\xi^4}{\sqrt{2}\xi^3} \sin \sqrt{2}\xi^3 s, \quad (3.6)$$

and for  $\xi^3 = 0$

$$\begin{aligned} x &= x_0 + \xi^1 s \\ y &= y_0 + \xi^2 s + \xi^1 \xi^4 s^2 \\ z &= z_0 \\ t &= t_0 + \xi^4 s \end{aligned} \quad (3.7)$$

To solve equations (2.9)- (2.12) for the case ( $\xi^3 \neq 0$ ) we have from equation (2.5)

$$\dot{x}(s) = 2\xi^3 t(s) + \xi^1 - 2\xi^3 t_0. \quad (3.8)$$

Solution of the equation (2.10) is:

$$Q = h_2 s + Q_0. \quad (3.9)$$

Using equations (3.5), (3.8) and (3.9) in equation (2.9) we have:

$$\begin{aligned} \ddot{P} &= 4Q\xi^3[(\xi^1 - \xi^3 t_0) \cos(\sqrt{2}\xi^3)s + \\ &\frac{\xi^4}{\sqrt{2}} \sin(\sqrt{2}\xi^3)s] + 4\xi^3 h_2(\xi^1 - \xi^3 t_0)s \cos(\sqrt{2}\xi^3)s \\ &+ 2\sqrt{2}\xi^3 \xi^4 h_2 s \sin(\sqrt{2}\xi^3)s - 2h_2 \dot{t} + 2\xi^3 \dot{V}, \end{aligned} \quad (3.10)$$

Integrating with respect to  $s$  we get:

$$\dot{P} = 2\sqrt{2}Q_0(\xi^1 - \xi^3 t_0) \sin(\sqrt{2}\xi^3)s - 2Q_0\xi^4 \cos(\sqrt{2}\xi^3)s +$$

$$\begin{aligned}
& 4\xi^3 h_2 (\xi^1 - \xi^3 t_0) \left[ \frac{s}{\sqrt{2}\xi^3} \sin(\sqrt{2}\xi^3)s + \frac{1}{2(\xi^3)^2} \cos(\sqrt{2}\xi^3)s \right] + \\
& 2\sqrt{2}\xi^3 \xi^4 h_2 \left[ \frac{-s}{\sqrt{2}\xi^3} \cos(\sqrt{2}\xi^3)s + \frac{1}{2(\xi^3)^2} \sin(\sqrt{2}\xi^3)s \right] \\
& - 2h_2 t(s) + 2\xi^3 V + C_1,
\end{aligned} \tag{3.11}$$

where  $C_1$  is a constant of integration. Using equation (3.6) in equation (3.11) we get:

$$\begin{aligned}
\dot{P} = & L_1 \sin(\sqrt{2}\xi^3)s + L_2 \cos(\sqrt{2}\xi^3)s + L_3 s \sin(\sqrt{2}\xi^3)s + \\
& L_4 s \cos(\sqrt{2}\xi^3)s + L_5 + 2\xi^3 V + C_1,
\end{aligned} \tag{3.12}$$

where

$$\begin{aligned}
L_1 = 2\sqrt{2}Q_0(\xi^1 - \xi^3 t_0), \quad L_2 = -2Q_0\xi^4, \quad L_3 = 2\sqrt{2}h_2(\xi^1 - \xi^3 t_0), \\
L_4 = -2\xi^4 h_2, \quad L_5 = \frac{2h_2\xi^1}{\xi^3} - 4h_2 t_0.
\end{aligned}$$

From the initial conditions (at  $s \rightarrow 0$  we have  $V = V_0$  and  $\dot{P} = h_1$ ) in equation (3.12) we get:

$$L_5 + C_1 = h_1 - L_2 - 2\xi^3 V_0,$$

then we have:

$$\begin{aligned}
\dot{P} = & L_1 \sin(\sqrt{2}\xi^3)s + L_2(\cos(\sqrt{2}\xi^3)s - 1) + L_3 s \sin(\sqrt{2}\xi^3)s + \\
& L_4 s \cos(\sqrt{2}\xi^3)s + 2\xi^3(V - V_0) + h_1,
\end{aligned} \tag{3.13}$$

using equations (3.5), (3.6), (3.9) and (3.13) in equation (2.12) we get:

$$\ddot{V} + 2(\xi^3)^2 V = M + M_1 \sin(\sqrt{2}\xi^3)s + M_2 \cos(\sqrt{2}\xi^3)s, \tag{3.14}$$

where

$$\begin{aligned}
M = 2\xi^1 h_2 - 2\xi^3(h_1 + 2\xi^4 Q_0 + 2h_2 t_0) + 4(\xi^3)^2 V_0, \\
M_1 = -2\sqrt{2}\xi^4 h_2, \quad M_2 = -4h_2(\xi^1 - \xi^3 t_0).
\end{aligned}$$

The solution of equation (3.14) is given by:

$$V(s) = A_1 \sin(\sqrt{2}\xi^3)s + A_2 \cos(\sqrt{2}\xi^3)s + \frac{M}{2(\xi^3)^2}$$

$$-\frac{M_1}{2\sqrt{2}\xi^3}s\cos(\sqrt{2}\xi^3)s + \frac{M_2}{2\sqrt{2}\xi^3}s\sin(\sqrt{2}\xi^3)s, \quad (3.15)$$

where  $A_1$  and  $A_2$  are constants.

From the initial condition  $s = 0$  we have:

$$V = V_0, \quad \dot{V} = h_4,$$

and from the equation (3.15) we get:

$$A_1 = \frac{h_4}{\sqrt{2}\xi^3} + \frac{M_1}{4(\xi^3)^2}, \quad A_2 = V_0 - \frac{M}{2(\xi^3)^2}$$

Hence equation (3.15) becomes :

$$\begin{aligned} V(s) = & K_1 \sin(\sqrt{2}\xi^3)s + K_2 \cos(\sqrt{2}\xi^3)s + K_3 s \sin(\sqrt{2}\xi^3)s \\ & + K_4 s \cos(\sqrt{2}\xi^3)s, \end{aligned} \quad (3.16)$$

where

$$K_1 = \frac{h_4}{\sqrt{2}\xi^3} + \frac{M_1}{4(\xi^3)^2}, \quad K_2 = V_0 - \frac{M}{2(\xi^3)^2}, \quad K_3 = \frac{M_2}{2\sqrt{2}\xi^3}, \quad K_4 = -\frac{M_1}{2\sqrt{2}\xi^3}.$$

Using the equation (3.16) in the equation (3.13) we get:

$$\dot{P} = H_1 \sin(\sqrt{2}\xi^3)s + H_2 \cos(\sqrt{2}\xi^3)s + H_3, \quad (3.17)$$

where

$$H_1 = \frac{1}{2\sqrt{2}\xi^3}[-\xi^4 h_2 + 4\xi^3(h_4 + 2Q_0(\xi^1 - \xi^3 t_0))], \quad H_2 = 2(h_1 - \frac{\xi^1 h_2}{\xi^3} + \xi^4 Q_0 + 2h_2 t_0 - \xi^3 V_0)$$

$$H_3 = -h_1 - 2\xi^4 Q_0 + \frac{2\xi^1 h_2}{\xi^3} - 4h_2 t_0 + 2\xi^3 V_0.$$

By integration with respect to  $s$  we obtain:

$$P = \frac{-H_1}{\sqrt{2}h_3} \cos(\sqrt{2}h_3)s + \frac{H_2}{\sqrt{2}h_3} \sin(\sqrt{2}h_3)s + H_3 s + C_2, \quad (3.18)$$

where  $C_2$  is a constant. From the initial condition at  $s = 0$  we have  $P = P_0$  then:

$$C_2 = P_0 + \frac{H_1}{\sqrt{2}\xi^2}.$$

Equation (3.18) read as:

$$P = P_0 + \frac{H_1}{\sqrt{2}\xi^3}(1 - \cos(\sqrt{2}\xi^3)s) + \frac{H_2}{\sqrt{2}\xi^3}\sin(\sqrt{2}\xi^3)s + H_3s. \quad (3.19)$$

In the following we integrate the equation for the coordinate  $U$ , using the equations (3.3), (3.5), (3.6), (3.8), (3.16) and (3.19) in the equation (2.11) we get:

$$\begin{aligned} \ddot{U} = & R_1 \sin(\sqrt{2}\xi^3)s + R_2 \cos(\sqrt{2}\xi^3)s + R_3 s \sin(\sqrt{2}\xi^3)s \\ & + R_4 s \cos(\sqrt{2}\xi^3)s + R_5 \sin^2(\sqrt{2}\xi^3)s \\ & + R_6 \cos^2(\sqrt{2}\xi^3)s + R_7 \sin(\sqrt{2}\xi^3)s \cos(\sqrt{2}\xi^3)s, \end{aligned} \quad (3.20)$$

where  $R_1, R_2, R_3, R_4, R_5, R_6$ , and  $R_7$  are constants read as:

$$\begin{aligned} R_1 = & \frac{1}{\sqrt{2}\xi^3} [8(\xi^1)^2 h_2 - (\xi^4)^2 h_2 + 4\xi^3 \xi^4 (h_4 + \xi^3 (P_0 + 4Q_0 t_0)) + 4(\xi^3)^2 t_0 (3h_1 + 6h_2 t_0 \\ & - 4\xi^3 V_0) + 4\xi^1 \xi^3 (-2h_1 - 2\xi^4 Q_0 - 7h_2 t_0 + 3\xi^3 V_0)], \\ R_2 = & \frac{1}{2\xi^3} [-7\xi^1 \xi^4 h_2 - 8(\xi^3)^3 t_0 (P_0 - 2Q_0 t_0) + 4\xi^3 (2(\xi^4)^2 Q_0 + \xi^1 (3h_4 + 4\xi^1 Q_0) + \xi^4 (h_1 + 3h_2 t_0)) \\ & + 8(\xi^3)^2 (-2h_4 t_0 + \xi^1 (P_0 - 4Q_0 t_0) - \xi^4 V_0)], \\ R_3 = & -2\sqrt{2}\xi^4 (-\xi^1 h_2 + \xi^3 (h_1 + 2\xi^4 Q_0 + 2h_2 t_0) - 2(\xi^3)^2 V_0), \\ R_4 = & 4(\xi^1 - \xi^3 t_0)(\xi^1 h_2 - \xi^3 (h_1 + 2\xi^4 Q_0 + 2h_2 t_0) + 2(\xi^3)^2 V_0) \\ R_5 = & \frac{1}{\xi^3} [-5\xi^1 \xi^4 h_2 + 8(\xi^3)^3 Q_0 t_0^2 + \xi^3 (4(\xi^4)^2 Q_0 + 4\xi^1 (h_4 + 2\xi^1 Q_0) + \xi^4 (4h_1 + 9h_2 t_0)) \\ & - 4(\xi^3)^2 (h_4 t_0 + 4\xi^1 Q_0 t_0 + \xi^4 V_0)], \\ R_6 = & \frac{1}{\xi^3} [5\xi^1 \xi^4 h_2 - 8(\xi^3)^3 Q_0 t_0^2 - \xi^3 (4(\xi^4)^2 Q_0 + 4\xi^1 (h_4 + 2\xi^1 Q_0) \\ & + \xi^4 (4h_1 + 9h_2 t_0)) + 4(\xi^3)^2 (h_4 t_0 + 4\xi^1 Q_0 t_0 + \xi^4 V_0)], \\ R_7 = & \frac{\sqrt{2}}{\xi^3} [(-8\xi^1)^2 h_2 + (\xi^4)^2 h_2 - 4\xi^3 \xi^4 h_4 + 8\xi^1 \xi^3 (h_1 + 3h_2 t_0 - \xi^3 V_0) + 8(\xi^3)^2 t_0 (-h_1 - 2h_2 t_0 + \xi^3 V_0)]. \end{aligned}$$

Integrating twice and using the initial conditions we get:

$$U(s) = U_0 + N_1 \sin(\sqrt{2}\xi^3)s + N_2 (\cos(\sqrt{2}\xi^3)s - 1)$$

$$\begin{aligned}
& +N_3 s \sin(\sqrt{2}\xi^3)s + N_4 s \cos(\sqrt{2}\xi^3)s + N_5 \sin(2\sqrt{2}\xi^3)s \\
& +N_6(\cos(2\sqrt{2}\xi^3)s - 1) + N_7 s,
\end{aligned} \tag{3.21}$$

where  $N_1, N_2, N_3, N_4, N_5, N_6$  and  $N_7$  are constants and are given by:

$$N_1 = \frac{1}{2\sqrt{2}(\xi^3)^3} [(\xi^4)^2 h_2 - 4\xi^3 \xi^4 (h_4 + \xi^3 P_0 + 2\xi^1 Q_0) + 4\xi^3 (\xi^1 h_2 t_0 + \xi^3 (-h_1 t_0 - 2h_2 t_0^2 + \xi^1 V_0))],$$

$$N_2 = \frac{1}{4(\xi^3)^3} [-\xi^1 \xi^4 h_2 + 8(\xi^3)^3 t_0 (P_0 - 2Q_0 t_0) + 4\xi^3 (2(\xi^4)^2 Q_0 - \xi^1 (3h_4 + 4\xi^1 Q_0)$$

$$+ \xi^4 (h_1 + h_2 t_0)) - 8(\xi^3)^2 (-2h_4 t_0 + \xi^1 (P_0 - 4Q_0 t_0) + \xi^4 V_0)],$$

$$N_3 = \frac{\sqrt{2}\xi^4 (-\xi^1 h_2 + \xi^3 (h_1 + 2\xi^4 Q_0 + 2h_2 t_0) - 2(\xi^3)^2 V_0)}{(\xi^3)^2},$$

$$N_4 = -\frac{2(\xi^1 - \xi^3 t_0)(\xi^1 h_2 - \xi^3 (h_1 + 2\xi^4 Q_0 + 2h_2 t_0) + 2(\xi^3)^2 V_0)}{(\xi^3)^2},$$

$$N_5 = \frac{1}{8\sqrt{2}(\xi^3)^3} [8(\xi^1)^2 h_2 - (\xi^4)^2 h_2 + 4\xi^3 \xi^4 h_4 + 8(\xi^3)^2 t_0 (h_1 + 2h_2 t_0 - \xi^3 V_0)$$

$$+ 8\xi^1 \xi^3 (-h_1 - 3h_2 t_0 + \xi^3 V_0)],$$

$$N_6 = \frac{1}{8(\xi^3)^3} [-5\xi^1 \xi^4 h_2 + 8(\xi^3)^3 Q_0 t_0^2 + \xi^3 (4(\xi^4)^2 Q_0 + 4\xi^1 (h_4 + 2\xi^1 Q_0) + \xi^4 (4h_1 + 9h_2 t_0))$$

$$- 4(\xi^3)^2 (h_4 t_0 + 4\xi^1 Q_0 t_0 + \xi^4 V_0)],$$

and

$$N_7 = -\frac{(\xi^4)^2 h_2}{4(\xi^3)^2} + h_3 + 2\xi^4 P_0 + 2h_1 t_0 + 4\xi^4 Q_0 t_0 + 4h_2 t_0^2 + \frac{\xi^4 h_4 - 2\xi^1 h_2 t_0}{\xi^3} - 2\xi^3 t_0 V_0,$$

For the second case ( $\xi^3 = 0$ ) it is easy to find:

$$\begin{aligned}
P &= P_0 + h_1 s - \xi^4 h_2 s^2 \\
Q &= Q_0 + h_2 s \\
U &= U_0 - (\xi^4 h_1 - \xi^1 h_4) s^2 + \frac{2}{3} h_2 [(\xi^4)^2 - (\xi^1)^2] s^3 + h_3 s \\
V &= V_0 + h_4 s - \xi^1 h_2 s^2.
\end{aligned} \tag{3.22}$$

These solutions for  $\xi^3 \neq 0$  and  $\xi^3 = 0$  satisfy also the condition given by equation (3.2).

Because of the homogeneity of the basic manifold ( also extended manifold)



without loss of generality, we consider the geodesics in eight dimension space time at the vertex (i.e  $x_0^i = 0, \Psi_0 = 0$ ).

For ( $\xi^3 \neq 0$ ) the equations (3.9), (3.16), (3.19) and (3.21) are reduced to:

$$P = \frac{1}{4(\xi^3)^2} [-4s(\xi^3)^2 h_1 + (-4\sqrt{2}\xi^1 \sin \sqrt{2}s\xi^3 + \xi^4(-1 + \cos \sqrt{2}s\xi^3))h_2 + 4\xi^3(\sqrt{2}h_1 \sin \sqrt{2}s\xi^3 + 2s\xi^1 h_2 + h_4 - h_4 \cos \sqrt{2}s\xi^3)] \quad (3.23)$$

$$Q = h_2 s \quad (3.24)$$

$$U = \frac{1}{16(\xi^3)^3} [\sqrt{2}(-8\xi^1 \xi^3 h_1 + 8(\xi^1)^2 h_2 - (\xi^4)^2 h_2 + 4\xi^3 \xi^4 h_4) \sin 2\sqrt{2}s\xi^3 - 4\sqrt{2}(-4s(\xi^3)^2 \xi^4 h_1 - (\xi^4)^2 h_2 + 4\xi^3 \xi^4 (s\xi^1 h_2 + h_4)) \sin \sqrt{2}s\xi^3 + 2(7\xi^1 \xi^4 h_2 + 8s(\xi^3)^3 h_3 + 8s(\xi^3)^2 \xi^4 h_4 - \xi^3(\xi^4(12h_1 + 2s\xi^4 h_2) - 20\xi^1 h_4)) + 2(-5\xi^1 \xi^4 h_2 + \xi^3(4\xi^4 h_1 + 4\xi^1 h_4)) \cos 2\sqrt{2}s\xi^3 - 4(-4\xi^3 \xi^4 h_1 + 8s(\xi^1)^2 \xi^3 h_2 + \xi^1(\xi^4 h_2 + 4\xi^3(-2s\xi^3 h_1 + 3h_4))) \cos \sqrt{2}s\xi^3] \quad (3.25)$$

$$V(s) = \frac{1}{8(\xi^3)^2} [-8\xi^3 h_1 + 8\xi^1 h_2 + 8(-\xi^1 h_2 + \xi^3(h_1 + s\xi^4 h_2)) \cos \sqrt{2}s\xi^3 + \sqrt{2}(-8s\xi^1 \xi^3 h_2 - \xi^4 h_2 + 4\xi^3 h_4) \sin \sqrt{2}s\xi^3]. \quad (3.26)$$

Equation (3.2) lead to:

$$\hat{g}_{ij} \frac{dy^i}{ds} \frac{dy^j}{ds} \big|_{y^i=0} = 2\xi^1 h_1 + 2\xi^2 h_2 + \frac{3(\xi^4)^2 h_2}{2\xi^3} + 2\xi^3 h_3 + 2\xi^4 h_4 = \epsilon. \quad (3.27)$$

For  $\xi^3 = 0$  the equation (3.2) take the form:

$$\xi^1 h_1 + \xi^2 h_2 + \xi^4 h_4 = \epsilon, \quad (3.28)$$

also the equation (3.22) reduced to

$$\begin{aligned} P &= h_1 s - \xi^4 h_2 s^2, \\ Q &= h_2 s, \\ U &= -(\xi^4 h_1 - \xi^1 h_4) s^2 + \frac{2}{3} h_2 [(\xi^4)^2 - (\xi^1)^2] s^3 + h_3 s, \\ V &= h_4 s - \xi^1 h_2 s^2. \end{aligned} \quad (3.29)$$

The geometrical and physical properties of the geodesic equation, and the relation between the extended space and the basic space will be done in the future work.

## 4 Translation Surfaces

Translation surfaces in an arbitrary Riemannian space are defined by the systems of equations [10], [11]

$$\frac{\partial^2 x^i(u, v)}{\partial u \partial v} + \Gamma_{jk}^i \frac{\partial x^j(u, v)}{\partial u} \frac{\partial x^k(u, v)}{\partial v} = 0, \quad (4.1)$$

where  $\Gamma_{jk}^i$  is the christoffel symbols of the second kind.

In the following we study the translation surfaces of the Anti- Mach space time metric given by the equation (2.1). Using equation (2.1) in equation (4.1) we get:

$$\frac{\partial^2 x}{\partial u \partial v} - \frac{\partial z}{\partial u} \frac{\partial t}{\partial v} - \frac{\partial t}{\partial u} \frac{\partial z}{\partial v} = 0 \quad (4.2)$$

$$\frac{\partial^2 y}{\partial u \partial v} - \frac{\partial x}{\partial u} \frac{\partial t}{\partial v} - \frac{\partial t}{\partial u} \frac{\partial x}{\partial v} = 0 \quad (4.3)$$

$$\frac{\partial^2 z}{\partial u \partial v} = 0 \quad (4.4)$$

$$\frac{\partial^2 t}{\partial u \partial v} + \frac{\partial x}{\partial u} \frac{\partial z}{\partial v} + \frac{\partial z}{\partial u} \frac{\partial x}{\partial v} - 2t \frac{\partial z}{\partial u} \frac{\partial z}{\partial v} = 0. \quad (4.5)$$

The solution of the equation (4.4) takes the form

$$z(u, v) = f(u) + g(v). \quad (4.6)$$

Taking the function  $t(u, v)$  as in the following

$$t(u, v) = f(u) - g(v). \quad (4.7)$$

Using equations (4.6) and (4.7) in equation (4.2) we have the representation

$$x(u, v) = x_1(u) + x_2(v). \quad (4.8)$$

From the equations (4.8) and (4.2) we get:

$$\frac{x_1' - 2ff'}{f'} = \frac{x_2' + 2gg'}{g'} = C_3, \quad (4.9)$$

where  $C_3$  is a constant,  $f' = \frac{df(u)}{du}$  and  $g' = \frac{dg(v)}{dv}$ .

By integration we get

$$x_1 = f^2 + C_3 f + C_4, \quad x_2 = -g^2 - C_3 g + C_5, \quad (4.10)$$

where  $C_4$  and  $C_5$  are constants of integration.

From the equations (4.7), (4.8) and (4.10) in the equation (4.3) we get:

$$\frac{\partial^2 y}{\partial u \partial v} = -2(C_3 f' g' + f f' g' + g g' f'). \quad (4.11)$$

Solution of the equation (4.11) is given by:

$$y(u, v) = -2C_3 f g - f^2 g - g^2 f + G_1(u) + G_2(v) \quad (4.12)$$

where  $G_1(u)$  and  $G_2(v)$  are two arbitrary functions. It is obvious from equation (4.12) that  $y$  is not a sum of two functions, function depended on the coordinate  $u$  and the other depended on the coordinate  $v$ . Hence, there is no existence of a translation surface in four dimension [11], but its projection in the 3- dimensional space  $(x, z, t)$  is a translation surface and we have the following theorem.

**Theorem:** *For the anti-Mach metric: There exists a translation surface in the orthogonal projection in the 3-dimensional space  $(x, z, t)$  and the existence comprises up to two an arbitrary functions in one variable.*

## 5 Conclusion

We have offered a Riemann extension for the anti-Mach space time. It is found that the vacuum field equations in eight dimensional space time are also satisfied . The full geodesic equations in this extended space time are obtained. We studding also the translation surfaces for the basic space.

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